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REVIEW ARTICLE

INVERSE PROBLEMS FORMULATED IN TERMS OF FIRST-KIND FREDHOLM INTEGRAL EQUATIONS IN INDIRECT MEASUREMENTS

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Abstract

Direct measurements of many properties of real-world systems are not possible. Information on these properties can only be inferred from the result of measurements of other quantities which may be measured directly. The process comprising direct measurements of certain characteristics of the object followed by inference on its sought-for properties from the directly measured characteristics based on a mathematical relation between unknown properties and measured characteristics is called indirect measurement, whereas inference is referred to as an inverse problem in indirect measurement.

In general an inverse problem consists either in determining the characteristics of a system under study, driven by controlled or known exciting signals, or in reconstructing exciting signals acting on a system whose internal characteristics are known. In both cases, it is formulated in terms of a mathematical model relating unknown and measured characteristics and signals. One can distinguish continuous and discrete inverse problems, depending on whether the measured and sought-for quantities are represented by functions or by vectors (tuples), respectively. Very many nontrivial inverse problems in indirect measurements are ill-posed which means that they have no solution or the solution exists but is non-unique or unstable, *i.e.* very small disturbances in the measurement data result in large disturbances in the result of inference. High error amplification is referred to as ill-conditioning. Ill-posedness and ill-conditioning result from the lack of information on sought-for quantities, carried by the measurement data. Therefore, *a priori* knowledge about the space of admissible solutions has to be employed for solving such inverse problems.

The theory of inverse problems and – in particular – effective numerical methods for solving them are of great importance for measurement science and technology; they are crucial for the development of many measurement, imaging and diagnostic techniques. Indirect measurements may be formulated using various mathematical models of the measurement object followed by a measuring system. A broad class of inverse problems, being of importance for indirect measurements, is formulated in terms of Fredholm integral equations of the first kind. These problems are ill-posed and strongly ill-conditioned after discretization. Therefore, sophisticated inverse procedures, utilizing various kinds of *a priori* knowledge, are applied for solving them.

In this paper, theoretical and numerical aspects of inverse problem in indirect measurements are reviewed. In particular the concept of generalized solution (pseudosolution) and the notion of well-posedness is presented and analysed. The review is focused on inverse problems formulated in terms of Fredholm integral equations of the first kind: a general presentation of such problems, at the level of functional analysis, is followed by an overview of numerical aspects of their discretized versions. A concise presentation of selected groups of numerical methods, called inverse methods, for solving inverse problems is also provided.

Keywords: inverse problem, indirect measurements, ill-posedness, ill-conditioning, Fredholm integral equation of the first kind, inverse procedures, *a priori* information, regularization, Bayesian inferring.

1.Introduction

1.1. Motivation behind the review paper

The review paper aims at presenting inverse problems in indirect measurements, focusing on inverse problems formulated in terms of Fredholm integral equations of the first kind.

The inverse problem is explored in a generic way apart from any specific measurement technique or measurement system, although examples of important measurement, imaging and diagnostic techniques, based on inverse problem formulated in terms of Fredholm integral equation of the first kind, are briefly discussed. In the paper general properties, such as ill-posedness and ill-conditioning of this class of inverse problems, are investigated and numerical methods of solving their discretized version are discussed.

The review paper is motivated by the increasing importance of the inverse problems in indirect measurements formulated in terms of Fredholm integral equation of the first kind for measurement science and technology. Therefore, comprehending the generic theory of these problems and mastering sophisticated and effective methods of solving them are crucial for development and implementation of many modern measurement techniques and systems. The paper provides the review of relevant extensive bibliography from many areas and aims at presenting the fundamental concepts in a generic, not area-specific way.

1.2. Overview of the paper

The review paper is structured as follows. At the beginning (Section 2) it presents concisely the basic concepts of inverse problems in indirect measurements. In Section 3 the general mathematical formulation of the inverse problem in indirect measurement, the concept of generalized solution, also known as quasisolution, and the notion of well-posedness are discussed. In the following sections the paper focuses on the inverse problem formulated in terms of Fredholm integral equation of the first kind. In Section 4 the Fredholm problem is explored at the level of functional analysis and an insight into its ill-posedness is provided. In Section 5 the discretized form of the inverse problem of Fredholm type is presented and its ill-conditioning is investigated. Finally, Section 6 provides a concise presentation of selected groups of numerical inverse procedures. Section 7 presents the most important conclusions of the review paper.

1.3. Mathematical notation

The following notation convention is used throughout the review:

- x, y, … are quantities considered in generic way, *i.e.* either functions or vectors or also real-valued scalar variables;
- $\mathbf{x}$, $\mathbf{y}$, … are vectors of real-valued variables;
- x_i, y_i … are elements of vectors $\mathbf{x}$, $\mathbf{y}$;
- $\mathbf{X}$, $\mathbf{Y}$, … are matrices of real-valued variables;
- x_{ij}, y_{ij}, … are elements of the matrices $\mathbf{X}$, $\mathbf{Y}$;
- $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, … are estimates of the vectors $\mathbf{x}$, $\mathbf{y}$; …;
- $\widetilde{\mathbf{x}}$, $\widetilde{\mathbf{y}}$, … are noisy versions of the vectors $\mathbf{x}$, $\mathbf{y}$; …;

All the abbreviations and symbols used throughout this review are defined at their first appearance.

2. Basic concepts of inverse problems in indirect measurements

2.1. Inverse problems in indirect measurements

Direct measurements of many properties of real-world complex systems are not possible because of lack of appropriate measurement elements, particularly sensors and transducers, impossibility of locating elements of measurement system in appropriate places of the system or due to ethical reasons. These properties can only be inferred from the results of measurements of other quantities describing the system under study which may be measured directly. Realizing a direct measurement of certain characteristics of the object, followed by inference on its sought-for properties from the directly measured characteristics based on the mathematical relation between unknown properties and measured characteristics, is called indirect measurement [1]. The inference process is referred to as an inverse problem in indirect measurement.

Generally speaking, the quantities inferred in the indirect measurements are of one of two types:

- characteristics of the system under study represented by some functions, operators or a set of parameters in the case the system is driven by a controlled or known exciting signal;
- exciting signals acting on the system whose internal characteristics are known.

In this connection, the inverse problem in the first case can be defined as determining characteristics of the system or in the special case when the sought-for characteristics are represented by parameters – as parameter estimation. The inverse problem in the second case can be defined as signal reconstruction. Both kinds of inverse problem require that the structure of the relation between the sought-for quantities and the directly measured quantities is known. This relation is called a mathematical model of indirect measurements.

The notion of indirect measurement is deeply rooted in measurement science and technology and therefore it is used throughout the present review paper. However, it is necessary to mention that this concept was removed from the newest version of the "International vocabulary of metrology – basic and general concepts and associated terms" [2] due to the lack of fundamental distinction between indirect and nontrivial direct measurements. Within the paper the term "indirect measurement" refers to the process comprising direct measurement of certain quantity followed by explicit computations of sought-for quantity based on results of direct measurements.

The mathematical model of indirect measurements very frequently is thought of as an operator mapping sought-for quantities into directly measured quantities. In the typical case this operator is assumed to be known, however inverse problems are considered where the operator represents the sought-for characteristics or the operator is unknown together with the sought-for quantity. One can distinguish continuous and discrete inverse problems. In the typical case, when the operator in the mathematical model of indirect measurement is assumed to be known, the directly measured and sought-for quantities are represented by functions for continuous inverse problem and by vectors for discrete inverse problem. The discrete inverse problems arise in parameter estimation or as a result of discretization of continuous inverse problem.

The mathematical model of indirect measurements can assume various forms. Very important from the application point of view are continuous models formulated in terms of integral operators both nonlinear such as Wiener operator or Hammerstein operator and linear such as Fredholm operator or Volterra operator. Discretization of continuous models expressed by linear operators yields systems of linear algebraic equations. The problem of parameter estimation which is an example of the discrete inverse problem is also expressed by

a system of linear algebraic equations. Mathematical modelling of measurement systems and signals is discussed in [3] and [4].

Inference of unknown quantities from the results of direct measurements of some other quantities is based on the information on sought-for quantities carried by the measurement data. Unfortunately, in the case of numerous nontrivial inverse problems in indirect measurements this information is insufficient for determining sought-for quantities in a unique way and with satisfactory accuracy. In other words, the condition of agreement of the sought-for solution with measurement data within the framework of given measurement uncertainty, understood as satisfying by the sought-for solution the mathematical model relation for these measurement data, does not determine this solution uniquely and accurately enough. It manifests itself in a very wide uncertainty interval of the solution even for very narrow uncertainty interval of the measurement data. This is very often called measurement error amplification or sensitivity to measurement error because even very small disturbances of the measurement data result in large disturbances of inferred corresponding sought-for quantities. This problem affecting the solution of an inverse problem is referred to as instability. The inverse problem which does not have any solution or the solution exists but is non-unique or is unstable is called an ill-posed problem.

Because the ill-posedness results from deficiency of information on sought-for quantities carried by measurement data in order to determine with satisfactory accuracy the unique solution of the inverse problem, additional *a priori* information on sought-for quantities is needed. Unstable problems can be thought of as quasi-underdetermined problems since the condition of agreement of the sought-for quantity with measurement data expressed by the mathematical model determines a very wide range of solutions. Therefore the *a priori* information on the sought-for solution very frequently assumes the form of additional constraining conditions imposed on the sought-for solution which enable to choose one "best" solution from the multitude of solutions.

Many techniques of solving ill-posed inverse problems called inverse methods, techniques, procedures or *a*lgorithms exist that apply various forms of *a priori* information.

The inverse problem in indirect measurements can be explored from various perspectives:

- topology and functional analysis [5];
- numerical analysis;
- statistics – statistical inference, estimation theory and information theory [1, 6].

This branch of knowledge is also strongly related to signal and image processing [7, 8]. Moreover, it makes use of methods of artificial intelligence such as neural networks and genetic algorithms [9-12] as well as of classical and modern optimization techniques [6].

Inverse problems theory and in particular development of effective inverse techniques are of great importance for the measurement science and technology. Some of modern measurement, imaging or diagnostic techniques based on indirect measurements, such as imaging of the interior of the objects based on measurements performed on their surfaces or in remote exterior region (*e.g.* various types of tomography – optical, microwave, X-ray and acoustic tomography utilized in medicine, engineering and Earth sciences) or remote sensing, would not be feasible at all without efficient, sophisticated methods developed by the inverse problems theory. Many measurement techniques which were considered to be not realizable because of ill-posedness are being implemented thanks to new inverse algorithms addressing ill-posedness. The importance of the inverse problem in indirect measurements is proved by the great number of applications in various branches of measurement technology, *e.g.* in measuring road traffic parameters [13], weighing moving vehicles [14, 15], measuring the parameters of optical elements [16], identification of models of electronic elements and systems and estimation of parameters of these models [17, 18] and others [19].

2.2. Inverse problems formulated in terms of Fredholm integral equations of the first kind

A broad class of continuous linear inverse problems being of importance for measurement science and technology, considerable practical meaning is formulated in terms of Fredholm integral equations of the first kind.

Inverse problems of this type arise in many areas, such as:

- computerized transmission tomography utilized in medicine, engineering and science consisting in determining the spatial distribution of a certain physical quantity inside the object under study and thereby the internal structure of this object, based on measurements of transmission of some physical excitation like radiation, acoustic (or in particular seismic) waves or electrical current [5, 20, 21];
- geophysical researches (including exploration of mineral deposits) such as [20]:
  - earthquake location and determination of acoustic wave velocity structure of the Earth from travel time data;
  - determination of velocity structure of the Earth from seismic surface waves;
  - determination of attenuation structure of the Earth from seismic waves;
  - determination of the density and magnetization structure of the Earth from gravimetric and geomagnetic data respectively;
  - remote sensing of the atmosphere: determining the temperature profile from remote radiance measurements, determining the composition from radiometric and spectrophotometric measurements as well as determining a particulate structure from light scattering measurements [22];
- spectrophotometric researches applied in qualitative and quantitative analysis of the composition of chemical substances [23-25];
- particle sizing, *i.e.* determining particle size distribution of dispersed phase of dispersed systems based on various physical properties of dispersed systems such as aerodynamic, electrical, diffusion and optical properties [6, 9-12, 26-32] which is the subject of research conducted by authors [33-40];
- digital signal processing, including images processing, within the problems of signal reconstruction and denoising as well as correction of static and dynamic characteristics of measurement sensors and transducers.

It can be proved that the inverse problem formulated in terms of Fredholm integral equation of the first kind is unstable and thereby an ill-posed problem. Hence, solving this problem requires using inverse techniques which apply *a priori* knowledge about the sought-for solution. In order to enable numerical solving the problem needs to be discretized *i.e.* continuous functions need to be represented by the vectors (tuples) of values. Discretization of the Fredholm integral equation of the first kind yields a system of linear algebraic equations. Ill-posedness of the continuous problem results in numerical ill-conditioning of the system of linear algebraic equations.

2.3. Numerical methods for solving inverse problems formulated in terms of Fredholm integral equation of the first kind

Many numerical inverse procedures were elaborated for solving discretized inverse problem formulated in terms of Fredholm integral equation of the first kind. Each of these methods applies a specific form of *a priori* information addressing ill-posedness and ill-conditioning. One can distinguish several groups of the inverse techniques: linear methods, nonlinear iterative methods, mixed (linear-nonlinear) methods, methods utilizing artificial neural networks and methods based on Bayesian statistical inference.

3. Mathematical formulation of inverse problems in indirect measurements

3.1. General mathematical model of indirect measurements

The inverse problem in indirect measurement consists in determining certain characteristics of the object under study represented by the quantity *g*, further on called sought-for quantity, on the basis of another quantity *h* obtained as a result of direct measurements and called measurement data, assuming known mapping *K* between the quantity *g* and the quantity *h*:

$$h = K(g) = Kg\,. \tag{3.1.1}$$

Quantities *g* and *h* can be represented by:

- functions $g(x)$ and $h(y)$ respectively in the case of continuous inverse problem,
- vectors $\mathbf{g} \in R^q$ and $\mathbf{h} \in R^p$ respectively, where R^n – *n* dimensional Euclidean space, in the case of a discrete inverse problem.

Equation (3.1.1) is the general mathematical model of indirect measurements.

3.2. Generalized solution (quasisolution) of the inverse problem in indirect measurements

Let us assume generally that *g* belongs to metric space *G* and *h* belongs to metric space *H*. Then *K* is the operator mapping the space *G* into the space *H*.

As a result of measurement an approximation of *h* is obtained which is denoted $\tilde{h}$. Hence, $\tilde{h}$ in general does not belong to the image of the operator *K*. In this connection, the equation:

$$\tilde{h} = Kg \tag{3.2.1}$$

may not have the solution $g \in G$ for arbitrary $\tilde{h} \in H$ [41].

If the solution $g \in G$ does not exist for arbitrary $\tilde{h} \in H$, an approximate solution of the equation (3.2.1) can be found. For that purpose the concept of so called generalized solution (quasisolution) can be employed which is defined as the quantity $\hat{g} \in G$ for which a metric (distance) $\rho_H(K\hat{g}, \tilde{h})$ attains its greatest lower bound [41]:

$$\rho_H(K\hat{g}, \tilde{h}) = \inf_{g \in G} \rho_H(Kg, \tilde{h}). \tag{3.2.2}$$

In the case that equation (3.2.1) has a solution $g \in G$ in an usual sense it coincides with the generalized solution given by the expression (3.2.2) [41].

3.3. Well-posedness of the inverse problem

Even if the unique generalized solution $\hat{g} \in G$ exists for arbitrary $\tilde{h} \in H$ and can be expressed as:

$$\hat{g} = R\tilde{h}\,, \tag{3.3.1}$$

where *R* – operator, very frequently this solution is not stable in the spaces (G, H). The problem of determining the solution (generalized solution) is said to be stable on the pair of metric spaces (G, H) if for every positive number ε a positive number exists $\delta(\varepsilon)$ such that the inequality $\rho_H(\tilde{h}_1, \tilde{h}_2) \le \delta(\varepsilon)$ implies $\rho_G(\hat{g}_1, \hat{g}_2) \le \varepsilon$, where [41]:

$$\hat{g}_1 = R\tilde{h}_1, \tag{3.3.2}$$

$$\hat{g}_2 = R\tilde{h}_2 \tag{3.3.3}$$

and

$$\hat{g}_1 \in G,\ \hat{g}_2 \in G,\ \tilde{h}_1 \in H,\ \tilde{h}_2 \in H. \tag{3.3.4}$$

In other words stability of determining the solution of the problem (3.2.1) means that the solution given by the expression (3.3.1) depends continuously on the measurement data $\tilde{h}$.

Stability of the solution of the inverse problem gives contribution to a broader property called well-posedness. The problem of determining the solution (generalized solution) $\hat{g} \in G$ of the equation (3.2.1) based on the measurement data $\tilde{h} \in H$ is said to be well-posed on the pair of metric spaces (G, H) if the following three Hadamard conditions are satisfied [41-45]:

1. for every measurement data $\tilde{h} \in H$ a solution $\hat{g} \in G$ exists;
2. the solution is unique;
3. the problem is stable on the pair of metric spaces (G, H).

Inverse problems and other mathematical problems not satisfying the above-mentioned conditions are called ill-posed problems [41-44].

As it was mentioned above the equation (3.2.1) may not have the solution $\hat{g} \in G$ for arbitrary data $\tilde{h} \in H$ because $\tilde{h}$ may not belong to the image of the operator *K* due to being corrupted by measurement errors. In this case the condition 1 of well-posedness is violated. Moreover, even if the solution (generalized solution) exists, very frequently it is unstable in the spaces (G, H) which means that the condition 3 of well-posedness is violated. Thereby, a vast majority of non-trivial inverse problems in indirect measurements are ill-posed problems [41].

4. Inverse problem formulated in terms of Fredholm integral equation of the first kind

4.1. Formulation of the inverse problem in terms of Fredholm integral equation of the first kind

A broad class of inverse problems, being of great importance for indirect measurements, is formulated in terms of Fredholm integral equation of the first kind [41, 42, 46]:

$$h(y) = K[g(x)] = \int_{x_{\min}}^{x_{\max}} K(y,x)g(x)\mathrm{d}x, \qquad y_{\min} \le y \le y_{\max}. \tag{4.1.1}$$

This type of problems represents linear inverse problems. In this case the operator *K* from the general equation (3.1.1) assumes the special form of the integral operator of Fredholm type.

4.2. Ill-posedness of the inverse problem formulated in terms of Fredholm integral equation of the first kind

In the present section ill-posedness of the inverse problem formulated in terms of Fredholm integral equation of the first kind is demonstrated.

Let us assume that the solution $g(x)$ of the equation (4.1.1) belongs to the space G of functions continuous on the interval $[x_{\min}, x_{\max}]$ and that changes in the solution $g(x)$ are measured in the corresponding C-metric defined by [41]:

$$\rho_G(g_1, g_2) = \max_{x \in [x_{\min}, x_{\max}]} |g_1(x) - g_2(x)|. \tag{4.2.1}$$

Let us also assume that the left-hand member $h(y)$ of the equation (4.1.1) belongs to the space H of functions square-integrable on the interval $[y_{\min}, y_{\max}]$ and that changes in $h(y)$ are measured in the corresponding L_2-metric defined by [41]:

$$\rho_H(h_1, h_2) = \left\{ \int_{y_{\min}}^{y_{\max}} [h_1(y) - h_2(y)]^2 \, \mathrm{d}y \right\}^{1/2}. \tag{4.2.2}$$

Solution of the equation (4.1.1) in the usual sense is unstable on the pair of metric spaces (G, H). In order to prove it, let us denote:

$$h_1(y) = \int_{x_{\min}}^{x_{\max}} K(y, x) g_1(x) \mathrm{d}x. \tag{4.2.3}$$

It can be noted that:

$$g_2(x) = g_1(x) + A \sin \omega x \tag{4.2.4}$$

is a solution of equation (4.1.1) with left-hand member:

$$h_2(y) = h_1(y) + A \int_{x_{\min}}^{x_{\max}} K(y, x) \sin \omega x \mathrm{d}x. \tag{4.2.5}$$

According to the Eq. (4.2.1) and (4.2.4) the change in the solution of the equation (4.1.1) is given by the formula:

$$\rho_G(g_1, g_2) = \max_{x \in [x_{\min}, x_{\max}]} |g_1(x) - g_2(x)| = \max_{x \in [x_{\min}, x_{\max}]} |A \sin \omega x| = |A|. \tag{4.2.6}$$

According to the equation (4.2.2) and (4.2.5) the change in the left-hand member of the equation (4.1.1) is given by the equation:

$$\rho_H(h_1, h_2) = \left\{ \int_{y_{\min}}^{y_{\max}} [h_1(y) - h_2(y)]^2 \, \mathrm{d}y \right\}^{1/2} = |A| \left\{ \int_{y_{\min}}^{y_{\max}} \left[\int_{x_{\min}}^{x_{\max}} K(y, x) \sin \omega x \mathrm{d}x \right]^2 \mathrm{d}y \right\}^{1/2}. \tag{4.2.7}$$

Because if $\omega \to \infty$, $\rho_H(h_1, h_2) \to 0$, for any number A the change in the left-hand member measured by $\rho_H(h_1, h_2)$ can be made arbitrarily small without preventing the change in the corresponding solution measured by $\rho_G(g_1, g_2)$ from being arbitrarily great [41]. This proves that the solution of the equation (4.1.1) is unstable on the pair of metric spaces (G, H) and thereby the inverse problem formulated in terms of Fredholm integral equation of the first kind is ill-posed [6, 41, 42, 45, 46].

The same conclusion can be drawn having assumed that the solution $g(x)$ of the equation (4.1.1) belongs to the space G of functions square-integrable on the interval $[x_{\min}, x_{\max}]$ and that changes in the solution $g(x)$ are measured in the corresponding L_2-metric [41].

4.3. Causes of ill-posedness of the inverse problem formulated in terms of Fredholm integral equation of the first kind

Insight into causes of ill-posedness of the inverse problem formulated in terms of Fredholm integral equation of the first kind can be obtained by analyzing it at the level of functional analysis.

Let us assume that the function $g(x)$ in the equation (4.1.1) belongs to the Hilbert space H_1 being the space $L_2[x_{\min}, x_{\max}]$ of all functions square-integrable in the interval $[x_{\min}, x_{\max}]$ with scalar product defined by the formula:

$$\langle t(x), u(x) \rangle_{H_1} = \int_{x_{\min}}^{x_{\max}} t(x) u(x) \mathrm{d}x \,. \tag{4.3.1}$$

Let us assume that the function $h(y)$ in the equation (4.1.1) belongs to the Hilbert space H_2 being the space $L_2[y_{\min}, y_{\max}]$ of all functions square-integrable in the interval $[y_{\min}, y_{\max}]$ with scalar product defined by the formula:

$$\langle t(y), u(y) \rangle_{H_2} = \int_{y_{\min}}^{y_{\max}} t(y) u(y) \mathrm{d}y \,. \tag{4.3.2}$$

Additionally, let us assume that the kernel $K(y,x)$ of the integral equation (4.1.1) is square-integrable, *i.e.*:

$$\int_{y_{\min}}^{y_{\max}} \left(\int_{x_{\min}}^{x_{\max}} [K(y,x)]^2 \mathrm{d}x \right) \mathrm{d}y < \infty \tag{4.3.3}$$

and that the integral operator $K : H_1 \to H_2$ defined by the equation (4.1.1) is a compact operator [42]. Then functions $u_n(y)$ and $v_n(x)$ exist, called singular functions, and real numbers σ_n, called singular values, such that $\sigma_1 \geq \sigma_2 \geq \ldots > 0$, $\sigma_n \to 0$ when $n \to \infty$ and [42, 47]:

$$(K v_n)(y) = \int_{x_{\min}}^{x_{\max}} K(y,x) v_n(x) \mathrm{d}x = \sigma_n u_n(y), \tag{4.3.4}$$

$$(K^* u_n)(x) = \int_{y_{\min}}^{y_{\max}} K(y,x) u_n(y) \mathrm{d}y = \sigma_n v_n(x), \tag{4.3.5}$$

where K^* – operator adjoint to the operator K. Functions $v_n(x)$ form the orthonormal basis in the domain of the operator K – space H_1, whereas functions $u_n(y)$ form the orthonormal basis in the image of the operator K – space H_2. In this connection, the function $g(x) \in H_1$ and the function $h(y) \in H_2$ can be represented by the following expansions – Fourier series respectively:

$$g(x) = \sum_{n=1}^{\infty} g_n v_n(x), \tag{4.3.6}$$

$$h(y) = \sum_{n=1}^{\infty} h_n u_n(y) \tag{4.3.7}$$

where:

$$g_n = \langle g(x), v_n(x) \rangle_{H_1} = \int_{x_{\min}}^{x_{\max}} g(x) v_n(x) \mathrm{d}x \,, \tag{4.3.8}$$

$$h_n = \langle h(y), u_n(y) \rangle_{H_2} = \int_{y_{\min}}^{y_{\max}} h(y) u_n(y) \mathrm{d}y \,. \tag{4.3.9}$$

Substituting expressions (4.3.6) and (4.3.7) to the equation (4.1.1) and applying relation (4.3.4) one obtains [42, 47]:

$$h_n = \sigma_n g_n \,, \tag{4.3.10}$$

and consequently:

$$g(x) = \sum_{n=1}^{\infty} \frac{h_n}{\sigma_n} v_n(x) = \sum_{n=1}^{\infty} \frac{\langle h(y), u_n(y) \rangle_{H_2}}{\sigma_n} v_n(x) \,. \tag{4.3.11}$$

The expansion (4.3.11), called the Picard series, determines solution of the inverse problem (4.1.1). The condition of existence of this solution is convergence of the Picard series, which can be strictly expressed as [42]:

$$\sum_{n=1}^{\infty} \left| \frac{\langle h(y), u_n(y) \rangle_{H_2}}{\sigma_n} \right|^2 < \infty \,. \tag{4.3.12}$$

The condition (4.3.12) is referred to as the Picard criterion [42].

Very frequently the kernel function $K(y,x)$ in the Fredholm integral equation of the first kind (4.1.1), which expresses the inverse problem, is *"smooth"*, *i.e.* it varies relatively slowly with *x*. Hence, from a qualitative point of view the integral operator *K* causes *"*smoothing*"* of the function $g(x)$. More precisely, it can be expressed by two general properties frequently noticed in the case of smoothing integral operators [46]:

- the singular values σ_n decay gradually to zero with *n*;
- the singular functions $u_n(y)$ and $v_n(x)$ tend to have a more and more oscillatory nature, *i.e.* more and more sign changes.

Hence, Fourier expansions (4.3.6) and (4.3.7) are decompositions of the functions $g(x)$ and $h(y)$ respectively on orthogonal oscillatory components of frequencies increasing with *n*. According to the formula (4.3.10) the integral operator *K* can be treated as a filter which dampens high-frequency oscillations of the function $g(x)$ as a result of multiplication of coefficients g_n for greater *n* by very small singular values σ_n which yields very small coefficients h_n. Therefore, the measurement data $h(y)$ is almost insensitive to the high-frequency oscillations in the function $g(x)$ even if they have considerable magnitude. On the other hand, solving the inverse problem causes amplification of high-frequency oscillations [6, 46].

As a result of measurement an approximation of $h(y)$ is obtained which is denoted $\widetilde{h}(y)$. Assuming that measurement errors and round-off errors corrupting the function $h(y)$ are additive random errors represented by the stochastic process $e(y)$, one can note:

$$\widetilde{h}(y) = h(y) + e(y) \,. \tag{4.3.13}$$

Replacing the function $h(y)$ in the expression (4.3.11) by the function $\tilde{h}(y)$ given by the expression (4.3.13) yields the solution $\hat{g}(x)$ determined for measurement data $\tilde{h}(y)$ [42]:

$$\begin{aligned}\hat{g}(x) &= \sum_{n=1}^{\infty} \frac{\left\langle \tilde{h}(y), u_n(y) \right\rangle_{H_2}}{\sigma_n} v_n(x) = \sum_{n=1}^{\infty} \frac{\left\langle h(y) + e(y), u_n(y) \right\rangle_{H_2}}{\sigma_n} v_n(x) \\ &= \sum_{n=1}^{\infty} \frac{\left\langle h(y), u_n(y) \right\rangle_{H_2}}{\sigma_n} v_n(x) + \sum_{n=1}^{\infty} \frac{\left\langle e(y), u_n(y) \right\rangle_{H_2}}{\sigma_n} v_n(x) \\ &= g(x) + \sum_{n=1}^{\infty} \frac{\left\langle e(y), u_n(y) \right\rangle_{H_2}}{\sigma_n} v_n(x) = g(x) + r(x).\end{aligned} \tag{4.3.14}$$

The random noise $e(y)$ can be considered as a "wideband" signal in terms of Fourier expansion into functions $u_n(y)$, *i.e.* the coefficients $\left\langle e(y), u_n(y) \right\rangle_{H_2}$ of its expansion into functions $u_n(y)$ are approximately constant with *n*. On the contrary, the function $h(y)$ is smooth, *i.e.* the coefficients $\left\langle h(y), u_n(y) \right\rangle_{H_2}$ of its expansion into functions $u_n(y)$ decay quickly with *n*. As a result within the process of solving the inverse problem, the high-frequency components originating from the noise $e(y)$ are amplified the most because of dividing the coefficients $\left\langle e(y), u_n(y) \right\rangle_{H_2}$ by insignificant singular values σ_n for big *n*. Hence, the coefficients $\left\langle \tilde{h}(y), u_n(y) \right\rangle_{H_2} / \sigma_n$ of the expansion of the solution $\hat{g}(x)$ into functions $v_n(x)$ are close to the coefficients of the true solution $\left\langle h(y), u_n(y) \right\rangle_{H_2} / \sigma_n$ merely for small *n*. For big *n* these coefficients are prevailed by noise contribution $\left\langle e(y), u_n(y) \right\rangle_{H_2} / \sigma_n$, because the coefficients for the true solution $\left\langle h(y), u_n(y) \right\rangle_{H_2} / \sigma_n$ decay quickly to zero with *n*. This leads to a tremendous increase of the error of the solution $\|r(x)\|$ for even a small measurement error $\|e(y)\|$ which proves so called *ill-conditioning* of the inverse problem that is inherently related to the ill-posedness [42].

Formally, the condition of existence of the solution $\hat{g}(x)$ is the Picard criterion:

$$\sum_{n=1}^{\infty} \left| \frac{\left\langle \tilde{h}(y), u_n(y) \right\rangle_{H_2}}{\sigma_n} \right|^2 < \infty . \tag{4.3.15}$$

According to the criterion the Fourier coefficients $\left\langle \tilde{h}(y), u_n(y) \right\rangle_{H_2}$ need to vanish with *n* faster than the singular values σ_n [42]. Violating the condition (4.3.15) indicates ill-posedness, whereas the bigger the terms of the series (4.3.15) for big *n* the more ill-conditioned the problem is [42].

5. Discretized form of inverse problems formulated in terms of Fredholm integral equation of the first kind

5.1. Discretization of inverse problems formulated in terms of Fredholm integral equation of the first kind

In order to apply numerical inverse procedures the inverse problem formulated in terms of Fredholm integral equation of the first kind has to be discretized, *i.e.* functions $g(x)$ and $h(y)$ in the equation (4.1.1) have to be represented by the vectors (tuples) of values. Numerous methods of discretization of the equation (4.1.1) exist [6, 42].

The Galerkin method consists in approximating the function $g(x)$, which is assumed to belong to the infinite-dimensional space $L_2[x_{\min}, x_{\max}]$, by its orthogonal projection to the finite-dimensional subspace of the space $L_2[x_{\min}, x_{\max}]$ spanned by linearly independent functions $\varphi_j(x),\ j=1,\ldots,q$ [42]:

$$\hat{g}(x) \approx \sum_{j=1}^{q} g_j \varphi_j(x). \tag{5.1.1}$$

The solution of the problem is the function $\hat{g}(x)$ for which $K[\hat{g}(x)]$ and $h(y)$ have identical orthogonal projection to the finite-dimensional subspace of the space $L_2[y_{\min}, y_{\max}]$ spanned by linearly independent functions $\psi_i(y),\ i=1,\ldots,p$ [42]:

$$\langle (K\hat{g})(y), \psi_i(y) \rangle = \langle h(y), \psi_i(y) \rangle, \qquad i=1,\ldots,p. \tag{5.1.2}$$

Substituting the expression (5.1.1) to the equation (5.1.2) one obtains:

$$\left\langle K\sum_{j=1}^{q} g_j \varphi_j(x), \psi_i(y) \right\rangle = \sum_{j=1}^{q} g_j \langle (K\varphi_j)(y), \psi_i(y) \rangle = \langle h(y), \psi_i(y) \rangle, \quad i=1,\ldots,p. \tag{5.1.3}$$

Denoting:

$$h_i = \langle h(y), \psi_i(y) \rangle, \qquad i=1,\ldots,p \tag{5.1.4}$$

and

$$k_{ij} = \langle (K\varphi_j)(y), \psi_i(y) \rangle, \qquad i=1,\ldots,p,\ j=1,\ldots,q, \tag{5.1.5}$$

yields the equation [42]:

$$\mathbf{h} = \mathbf{K}\mathbf{g}. \tag{5.1.6}$$

The most straightforward choice of the functions $\psi_i(y)$ is:

$$\psi_i(y) = \delta(y - y_i), \quad y_{\min} \le y_i \le y_{\max}, \qquad i=1,\ldots,p \tag{5.1.7}$$

where $\delta(y)$ – Dirac distribution, y_i – measurement points. This leads to so called collocation method, which consists in representing the function $h(y)$ by the vector (tuple) of its values in collocation points y_i, corresponding to measurement points [42]:

$$h_i = h(y_i), \quad y_{\min} \le y_i \le y_{\max}, \quad i=1,\ldots,p. \tag{5.1.8}$$

Another way of discretizing the right-hand member of the equation (4.1.1) is to approximate the integration by a numerical quadrature, which consists in approximating the integral by the weighted sum of values of the function under integral [42]:

$$\int_{x_{\min}}^{x_{\max}} K(y,x)g(x)\mathrm{d}x \approx \sum_{j=1}^{q} w_j K(y,x_j)g(x_j) \tag{5.1.9}$$

where w_j – weighting coefficients. The points x_j and corresponding weights can be chosen in various ways.

In the midpoint rule the interval $[x_{\min}, x_{\max}]$ is divided into q equal subintervals, each of the length [42]:

$$\Delta x = \frac{x_{\max} - x_{\min}}{q}. \tag{5.1.10}$$

The j-th subinterval is [42]:

$$[x_{\min} + (j-1)\Delta x, x_{\min} + j\Delta x], \qquad j = 1, \ldots, q. \tag{5.1.11}$$

Each of the points x_j is the midpoint of the j-th subinterval [42]:

$$x_j = x_{\min} + \left(j - \frac{1}{2}\right)\Delta x. \tag{5.1.12}$$

The midpoint rule consists in approximating the function under integral by the piecewise constant function, which for consecutive subintervals assumes constant values equal to the values of the function under integral in the midpoints of the subintervals [42]:

$$\int_{x_{\min}}^{x_{\max}} K(y,x)g(x)\mathrm{d}x \approx \sum_{j=1}^{q} K(y,x_j)g(x_j)\Delta x. \tag{5.1.13}$$

In this connection for the midpoint rule [42]:

$$w_j = \Delta x. \tag{5.1.14}$$

In the trapezoid rule the function under integral is approximated by the piecewise linear function. The corresponding distribution of the x_j points is given by the formula [42]:

$$x_j = x_{\min} + (j-1)\Delta x, \tag{5.1.15}$$

where [42]:

$$\Delta x = \frac{x_{\max} - x_{\min}}{q-1}. \tag{5.1.16}$$

The weighting coefficients are determined by the expression [42]:

$$w_j = \begin{cases} \Delta x, & 2 \le j \le q-1 \\ \frac{1}{2}\Delta x, & j = 1, q. \end{cases} \tag{5.1.17}$$

Discretization of the equation (4.1.1) with use of the collocation method and numerical quadrature, based on equations (5.1.8) and (5.1.9) results in the following system of linear algebraic equations [42]:

$$h_i = h(y_i) = \sum_{j=1}^{q} w_j K(y_i, x_j) g(x_j), \ldots\ldots i = 1, \ldots, p \,. \tag{5.1.18}$$

Denoting:

$$g_j = g(x_j), \ldots\ldots j = 1, \ldots, q \tag{5.1.19}$$

and

$$k_{ij} = w_j K(y_i, x_j), \qquad i = 1, \ldots, p \,, \qquad j = 1, \ldots, q \,, \tag{5.1.20}$$

one obtains the equation (5.1.6) but with different meaning of the vectors and the matrix.

Thus, discretization of Fredholm integral equation of the first kind (4.1.1.) results in a system of algebraic linear equations of the form of expression (5.1.6) regardless of the method of discretization.

5.2. Ill-conditioning of discretized inverse problems formulated in terms of Fredholm integral equation of the first kind

The equation (5.1.6), which is the discretized form of Fredholm integral equation of the first kind, may be considered as a special case of the equation (3.1.1) where the operator K is a linear operator mapping Euclidean q-dimensional space G into another Euclidean space H. The operator K is represented by the matrix $\mathbf{K}$.

Similarly to the general case considered in Section 3.2, the measurement yields some approximation of the vector $\mathbf{h}$ denoted $\tilde{\mathbf{h}}$ which in general does not belong to the image of the operator K, *i.e.* to the column space of the matrix $\mathbf{K}$ [41]. Hence, the Eq.:

$$\tilde{\mathbf{h}} = \mathbf{K}\mathbf{g} \tag{5.2.1}$$

may not have the solution for arbitrary $\tilde{\mathbf{h}}$. However, the generalized solution (quasisolution) $\hat{\mathbf{g}}$ can be found in this case. The definition of the generalized solution is derived from the general definition (3.2.2) by assuming that the metric (distance) is represented by the Euclidean norm [41]:

$$\hat{\mathbf{g}} = \arg\min_{\mathbf{g}} \left\{ \left\| \tilde{\mathbf{h}} - \mathbf{K}\mathbf{g} \right\|_2 \right\}. \tag{5.2.2}$$

Vector $\mathbf{K}\hat{\mathbf{g}}$ is the orthogonal projection of the vector $\tilde{\mathbf{h}}$ onto the linear space spanned by columns of the matrix $\mathbf{K}$ and consequently $\hat{\mathbf{g}}$ is the representation of this projection in the basis formed by columns of the matrix $\mathbf{K}$. This can be expressed by the equation [48, 49]:

$$\mathbf{K}^T \left(\tilde{\mathbf{h}} - \mathbf{K}\hat{\mathbf{g}} \right) = 0 \,. \tag{5.2.3}$$

The solution $\hat{\mathbf{g}}$ is unique only if all columns of matrix $\mathbf{K}$ are linearly independent, *i.e.* if matrix $\mathbf{K}$ has full column rank [48, 49]:

$$r = q \le p \,. \tag{5.2.4}$$

Then the solution $\hat{\mathbf{g}}$ is given by the equation [48, 49]:

$$\hat{\mathbf{g}}_{\mathrm{OLS}} = \left(\mathbf{K}^T\mathbf{K}\right)^{-1}\mathbf{K}^T\tilde{\mathbf{h}} \tag{5.2.5}$$

and is called ordinary least squares (OLS) solution of the equation (5.2.1).

If the matrix **K** has not full column rank:

$$r < q\,, \tag{5.2.6}$$

infinitely many solutions $\hat{\mathbf{g}}$ exist which form $(q-r)$-dimensional manifold [48]. If certain vector $\hat{\mathbf{g}}_0$ is a solution of the equation (5.2.3), also a vector:

$$\hat{\mathbf{g}}_1 = \hat{\mathbf{g}}_0 + \hat{\mathbf{g}}_{\mathrm{null}}\,, \tag{5.2.7}$$

where:

$$\hat{\mathbf{g}}_{\mathrm{null}} \in \ker\mathbf{K}\,, \tag{5.2.8}$$

is a solution of the equation (5.2.3). However, from the multitude of solutions the one can be chosen uniquely which has the smallest Euclidean norm [48]:

$$\hat{\mathbf{g}}_{\mathrm{MNLS}} = \arg\min_{\hat{\mathbf{g}}}\left\{\left\|\hat{\mathbf{g}}\right\|_2\right\}. \tag{5.2.9}$$

The solution (5.2.9) is called the minimum norm least squares (MNLS) solution of the equation (5.2.1) or the pseudosolution of the equation (5.2.1). The vector $\hat{\mathbf{g}}_{\mathrm{MNLS}}$ is orthogonal to the kernel of the matrix **K**.

It can be shown that the solution $\hat{\mathbf{g}}_{\mathrm{MNLS}}$ is given by the expression [48, 49]:

$$\hat{\mathbf{g}}_{\mathrm{MNLS}} = \mathbf{K}^{+}\tilde{\mathbf{h}}\,, \tag{5.2.10}$$

where $\mathbf{K}^{+}$ denotes so called pseudoinverse of the matrix **K**, which is defined as the matrix satisfying the following conditions [49]:

1. $\mathbf{K}\mathbf{K}^{+}\mathbf{K} = \mathbf{K}$ (5.2.11a)
2. $\mathbf{K}^{+}\mathbf{K}\mathbf{K}^{+} = \mathbf{K}^{+}$ (5.2.11b)
3. $\mathbf{K}\mathbf{K}^{+} = \left(\mathbf{K}\mathbf{K}^{+}\right)^T$ (5.2.11c)
4. $\mathbf{K}^{+}\mathbf{K} = \left(\mathbf{K}^{+}\mathbf{K}\right)^T$. (5.2.11d)

The pseudoinverse can be determined based on the singular value decomposition (SVD) of the matrix **K** which is given by the equation [6, 42, 46, 48-50]:

$$\mathbf{K} = \mathbf{U}\mathbf{S}\mathbf{V}^T\,, \tag{5.2.12}$$

where: **U** – orthogonal matrix of size $p \times p$, **V** – orthogonal matrix of size $q \times q$, **S** – matrix of size $p \times q$ of the form [6, 42, 46, 48-50]:

$$\mathbf{S} = \begin{bmatrix} \mathbf{\Sigma} & \mathbf{0}_{r,q-r} \\ \mathbf{0}_{p-r,r} & \mathbf{0}_{p-r,q-r} \end{bmatrix}, \tag{5.2.13}$$

whereas: $\mathbf{\Sigma}$ – diagonal matrix of size $r \times r$, where r – the rank of the matrix **K**, $\mathbf{0}_{k,l}$ – matrix of size $k \times l$ containing only zeros. Diagonal elements of the matrix $\mathbf{\Sigma}$ called singular values

of the matrix **K** are positive and arranged in the decreasing order: $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$. Columns of the matrix **U** and **V** denoted $\mathbf{u}_n$, $n = 1, \ldots, p$ and $\mathbf{v}_n$, $n = 1, \ldots, q$, respectively, are singular vectors of the matrix **K**.

The pseudoinverse is given by the expression [49]:

$$\mathbf{K}^+ = \mathbf{V}\mathbf{S}^+\mathbf{U}^T, \tag{5.2.14}$$

where:

$$\mathbf{S}^+ = \begin{bmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0}_{r,p-r} \\ \mathbf{0}_{q-r,r} & \mathbf{0}_{q-r,p-r} \end{bmatrix}. \tag{5.2.15}$$

Substituting the expression (5.2.15) into (5.2.14) and then (5.2.14) into (5.2.10) one can obtain the following equation [6, 42, 46]:

$$\hat{\mathbf{g}}_{\text{MNLS}} = \sum_{n=1}^{r} \frac{\mathbf{u}_n{}^T \tilde{\mathbf{h}}}{\sigma_n} \mathbf{v}_n . \tag{5.2.16}$$

The formula (5.2.16) is the equivalent of the formula (4.3.11) for the discretized version of the inverse problem. It describes the expansion of the vector $\hat{\mathbf{g}}_{\text{MNLS}}$ into the Fourier series of orthonormal vectors $\mathbf{v}_n$, $n = 1, \ldots, r$. The expansion coefficients are corresponding coefficients of expansion of the vector $\tilde{\mathbf{h}}$ into the series of orthonormal vectors $\mathbf{u}_n$, $n = 1, \ldots, r$, given by expression $\mathbf{u}_i{}^T \mathbf{h}$, divided by corresponding singular values σ_n, $n = 1, \ldots, r$ [6, 42, 46].

As discussed in the Section 4.3, the cause of ill-posedness and related ill-conditioning of the inverse problem formulated in terms of Fredholm integral equation of the first kind is the fast rate of decaying the singular values σ_n of the integral operator with *n*, so that they approach zero for bigger *n*. This property is inherited by the discrete approximation of the integral operator which is represented by the matrix **K** in the equation (5.2.1) [6, 42, 46]. In this connection, the frequently observed and common property of the matrix **K** is that [46]:

- singular values σ_n, $n = 1, \ldots, r$ of the matrix **K** decay very quickly with *n* and approach zero for bigger *n*;
- the singular vectors $\mathbf{u}_n$ and $\mathbf{v}_n$ tend to have a more and more oscillatory nature, *i.e.* more and more sign changes.

Assuming that the vector $\tilde{\mathbf{h}}$ is corrupted by additive noise, being the resultant effect of measurement errors and round-off errors, one can write [6, 46]:

$$\tilde{\mathbf{h}} = \mathbf{h} + \mathbf{e}, \tag{5.2.17}$$

where **e** is a stochastic process representing the noise. Substituting the expression (5.2.17) to the equation (5.2.16) yields:

$$\begin{aligned} \hat{\mathbf{g}}_{\text{MNLS}} &= \sum_{n=1}^{r} \frac{\mathbf{u}_n{}^T \tilde{\mathbf{h}}}{\sigma_n} \mathbf{v}_n = \sum_{n=1}^{r} \frac{\mathbf{u}_n{}^T (\mathbf{h} + \mathbf{e})}{\sigma_n} \mathbf{v}_n = \sum_{n=1}^{r} \frac{\mathbf{u}_n{}^T \mathbf{h}}{\sigma_n} \mathbf{v}_n + \sum_{n=1}^{r} \frac{\mathbf{u}_n{}^T \mathbf{e}}{\sigma_n} \mathbf{v}_n \\ &= \mathbf{g}_{\text{MNLS}} + \mathbf{r}, \end{aligned} \tag{5.2.18}$$

where $\mathbf{g}_{\text{MNLS}}$ – the true minimum norm least squares solution, *i.e.* the minimum norm least squares solution for the case without the noise. The formula (5.2.18) is the equivalent of the formula (4.3.14) for the discretized version of the inverse problem. Like in the continuous

case, the coefficients $\mathbf{u}_n^T\mathbf{e}$ of the expansion of the vector $\mathbf{e}$ into vectors $\mathbf{u}_n$ are approximately constant for $n=1,\ldots,r$, whereas the coefficients $\mathbf{u}_n^T\mathbf{h}$ of the expansion of the vector $\mathbf{h}$ into vectors $\mathbf{u}_n$ decay quickly with $n=1,\ldots,r$, because of smoothness of the true measurement data $\mathbf{h}$. According to the equation (5.2.18) the higher-frequency terms of the expansion of the vector $\hat{\mathbf{g}}_{\mathrm{MNLS}}$ into vectors $\mathbf{v}_n$, i.e. the terms for bigger n, are dominated by the contribution from noise as a result of dividing coefficients $\mathbf{u}_n^T\mathbf{e}$ of significant values by very small singular values σ_n. Therefore, the error of the solution given by the term $\mathbf{r}$ in the formula (5.2.18) is tremendous in terms of Euclidean norm $\|\mathbf{r}\|_2$ and overwhelms the true solution $\mathbf{g}_{\mathrm{MNLS}}$ even for a small measurement error $\|\mathbf{e}\|_2$. This effect of amplification of measurement errors in the solution demonstrates ill-conditioning of the discretized inverse problem formulated in terms of Fredholm integral equation of the first kind [6, 42, 46].

The direct cause of ill-conditioning of the discretized version of the inverse problem is that the least singular values are very close to zero, which means that corresponding columns or rows of the matrix $\mathbf{K}$ are nearly linearly dependent [6, 41, 42, 46, 51]. Because the singular values of the matrix $\mathbf{K}$ are known only approximately they can be indistinguishable from zero within the framework of a given accuracy. This causes that the rank of the matrix $\mathbf{K}$ is not strictly defined [41, 46, 51]. The matrix $\mathbf{K}$ with the rank r is close to the matrix $\mathbf{K}'$ with the rank $r'<r$. In other words, the problem is nearly underdetermined.

Ill-conditioning causes that the uncertainty interval of the solution $\hat{\mathbf{g}}_{\mathrm{MNLS}}$ is very wide at even a narrow uncertainty interval of the data vector $\tilde{\mathbf{h}}$ [6]. Because of the very wide uncertainty interval of the solution $\hat{\mathbf{g}}_{\mathrm{MNLS}}$ this solution is quasi-nonunique. Hence, from the qualitative point of view the condition of agreement of the solution $\hat{\mathbf{g}}_{\mathrm{MNLS}}$ with the data vector $\tilde{\mathbf{h}}$ in terms of the criterion (5.2.2) as well as the condition of minimal norm (5.2.9) are insufficient to determine the solution $\hat{\mathbf{g}}_{\mathrm{MNLS}}$ with satisfactory accuracy, because a wide interval of the solutions satisfies these conditions within the framework of assumed accuracy [6]. In order to find a unique solution with satisfactory accuracy, an additional condition resulting from *a priori* knowledge about the sought-for solution needs to be imposed on the sought-for solution. Various forms of *a priori* conditions are applied for this purpose [5, 6].

In the Section 4.3 it was proved that the source of ill-posedness and ill-conditioning of the inverse problem formulated in terms of Fredholm integral equation of the first kind is the smoothing property of the integral operator. This property is reflected in the discretized version of the inverse problem and causes that unrealistic high-frequency oscillations are present in the minimum norm least squares solution. Hence, a natural *a priori* condition which can be imposed on the sought-for solution in order to counteract the effects of ill-conditioning is smoothness of the sought-for function $\hat{g}(x)$ and consequently smoothness of the vector $\hat{\mathbf{g}}_{\mathrm{MNLS}}$ which is the discrete representation of the function $\hat{g}(x)$ [6, 42, 46]. The condition of smoothness results also from the fact that in the inverse problem in indirect measurement the function $g(x)$ represents changes of some physical quantity with some variable and hence tends to be smooth. Various measures of lack of smoothness of the vector $\hat{\mathbf{g}}_{\mathrm{MNLS}}$ are applied. The most common ones are the Euclidean norms of the following vectors: the vector $\hat{\mathbf{g}}_{\mathrm{MNLS}}$, the vector of first, second and third order differences of the vector $\hat{\mathbf{g}}_{\mathrm{MNLS}}$, which are discrete approximations of the first, second and third derivative of the function $\hat{g}(x)$, respectively, and the difference of the vector $\hat{\mathbf{g}}_{\mathrm{MNLS}}$ and certain *a priori* assumed initial solution [6, 42, 46].

6. Selected numerical methods for solving inverse problems formulated in terms of Fredholm integral equation of the first kind

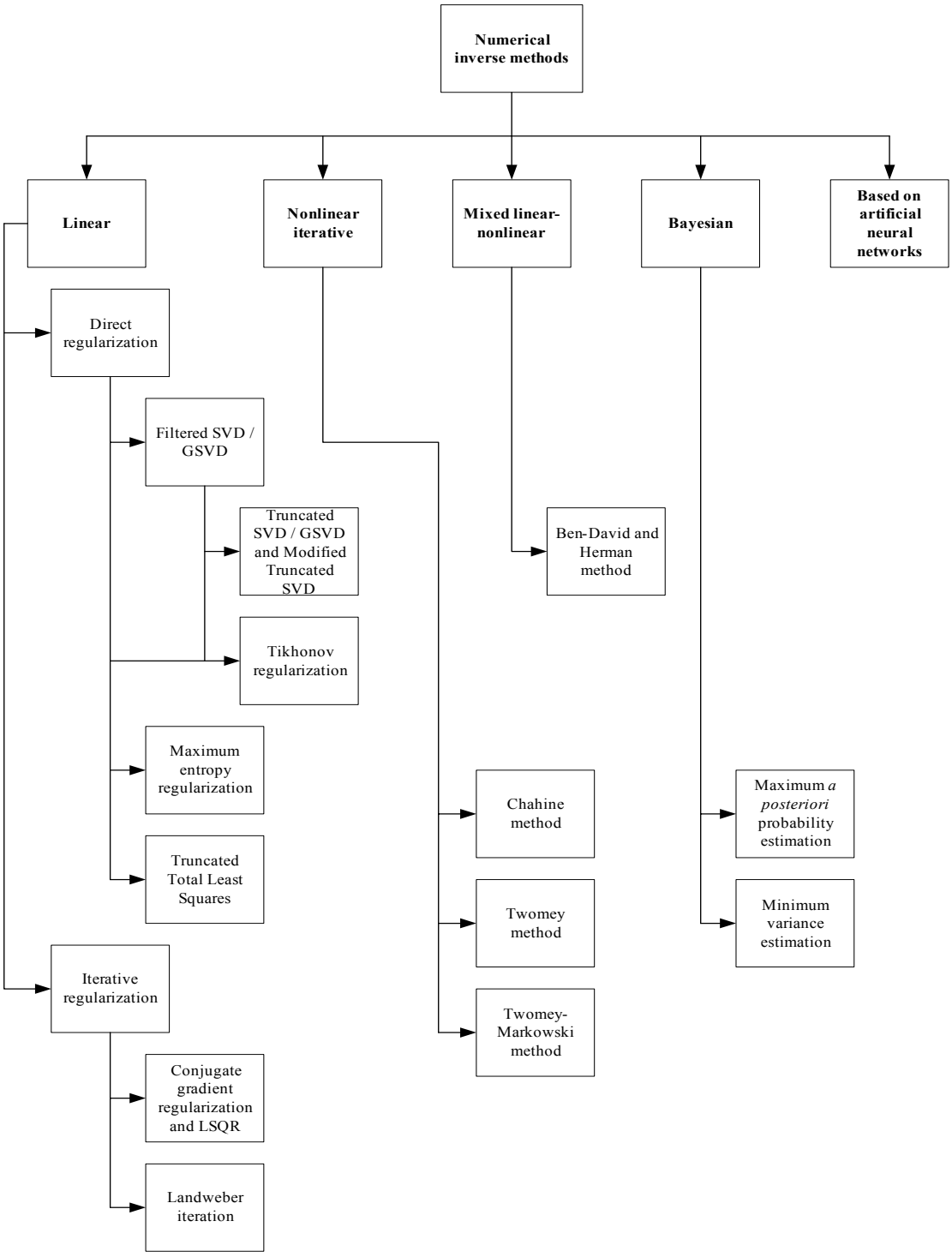

Fig. 6.1. General classification of the inverse methods with examples of techniques belonging to particular groups.

Many numerical inverse techniques were elaborated for solving the inverse problem formulated in terms of Fredholm integral equation of the first kind. Each of these procedures

utilizes a specific form of *a priori* knowledge about the sought-for solution in order to counteract the effects of ill-conditioning. The general classification of the inverse methods is presented in Fig. 6.1 [6, 42, 46].

The classification presented in Fig. 6.1 is not complete and it aims only at indicating the most important groups of numerical inverse techniques applied for solving the inverse problem formulated in terms of Fredholm integral equation of the first kind with the most significant representatives of particular groups.

The linear inverse methods consist in performing some linear operation (direct regularization) or sequence of operations (iterative regularization) on the vector $\tilde{\mathbf{h}}$ of measurement data in order to find the solution vector $\hat{\mathbf{g}}$. Direct regularization aims at finding the best approximation of the inverse operator which improves the numerical conditioning by applying *a priori* information on the sought-for solution [6, 28, 29, 31, 42, 46]. In general, regularization consists in converting the original ill-conditioned inverse problem into the new "nearby" inverse problem characterized by better conditioning, which results in decreasing the enormous uncertainty of the solution at the cost of deteriorating its agreement with the measurement data. Many direct regularization techniques can be formulated in singular value decomposition (SVD) or generalized singular value decomposition (GSVD) of the matrix **K** in the equation (5.2.1). In this approach regularization is realized by modifying the singular values or generalized singular values of the matrix **K** within the procedure called filtration [50, 52, 53]. Various schemes of filtration were elaborated including truncating the smallest singular values or generalized singular values [29, 54-56] and Tikhonov regularization, also known as Twomey-Phillips regularization [29, 41, 43, 57-59, 60]. Tikhonov regularization can be also formulated in terms of minimization of the sum of the term being the measure of discrepancy between the sought-for solution and measurement data and the term being the measure of deviation of the sought-for solution from *a priori* assumed characteristics [29, 41, 43, 57]. The truncated SVD procedure and Tikhonov regularization are also employed in the field of parameter estimation in statistics where they are called principal components regression (PCR) [61] and ridge regression, [62, 63] respectively. The direct regularization techniques are also represented by maximum entropy regularization based on the concept of entropy from information theory [6, 64] as well as by truncated total least squares (TTLS) method [65, 66] and *partial least squares (PLS)* method [67]. The iterative regularization techniques are represented by the conjugate gradients (CG) method [68] and by the *LSQR* method [69].

The key aspect of regularization is to determine the degree of regularization of the inverse problem, *i.e.* how much the *a priori* condition influences sought-for regularized solution compared to the condition of agreement of sought-for regularized solution with the measurement data. Increasing the degree of regularization improves the conditioning of the problem but at the same time causes that the regularized problem becomes more and more different from the original problem. This results in a smaller degree of uncertainty of the regularized solution at the cost of bigger deviation of this solution from the true solution, *i.e.* bigger bias or systematic error of the solution. The augmented degree of regularization increases the discrepancy between the solution and measurement data (more accurately – between the output of the mathematical model of measurements for particular solution and measurement data) called prediction error [70]. Hence, the optimal level of regularization has to be selected by means of trade-off between improving the numerical conditioning of the problem and keeping the discrepancy between regularized solution and measurement data within the acceptable range. In this connection, many numerical techniques were elaborated for this purpose. The most popular procedures are the discrepancy principle [6, 42, 44], generalized cross-validation *(GCV)* [6, 42, 71] and L-curve criterion [6, 42, 72]. Algorithms for truncated SVD method are described in [73].

Nonlinear iterative methods determine the solution by iterative corrections of an initial solution assumed *a priori*. These corrections are of nonlinear nature and their direction and magnitude is selected so that the solutions from consecutive iterations are characterized by better and better agreement with the measurement data. Therefore, the process is convergent to the true solution of the inverse problem [6]. The most popular representatives of this group of inverse procedures are: Chahine algorithm [29, 74], Twomey algorithm [75, 76] and its enhanced version – Twomey-Markowski algorithm [77].

Mixed linear-nonlinear techniques combine direct regularization with nonlinear iterative corrections. Within the Ben-David and Herman method an original discrete inverse problem is replaced by the discrete problem of reduced dimensionality formulated in terms of correction coefficients. The new problem is solved directly which yields a vector of correction coefficients. Subsequently this vector is used for nonlinear iterative corrections [6, 78].

Artificial neural networks *(ANN)* are also employed for solving inverse problems. An ANN is taught (by corrections of its weighting coefficients) the mathematical model of measurements (3.1.1), more specifically – the mapping of the vector $\mathbf{g}$ into the vector of measurement data $\mathbf{h}$. Then the taught ANN acts as an inverse model, *i.e.* when the vector $\tilde{\mathbf{h}}$ is passed to the input of the ANN the solution vector is obtained in the output of the ANN [29].

The Bayesian techniques, as distinct from other inverse procedures, do not solve the inverse problem explicitly. In these methods one assumes a certain *a priori* probability distribution of the sought-for vector $\mathbf{g}$. This distribution reflects the uncertainty of *a priori* knowledge about the sought-for solution of the problem. Subsequently, actual measurement data $\tilde{\mathbf{h}}$ are used for updating the *a priori* knowledge about the solution. It is done by determining the conditional probability distribution of the sought-for vector $\mathbf{g}$ on condition that the measurement yielded particular data $\tilde{\mathbf{h}}$. This resultant probability distribution, called a posteriori probability distribution, is probabilistic information on the solution. Based on this probability distribution one can determine various estimators of the solution [6, 64, 79]. Maximum a posteriori probability (MAP) estimator is the vector $\hat{\mathbf{g}}_{\mathrm{MAP}}$ for which a posteriori probability attains its maximum [6, 64, 79]. Minimum variance Bayesian (MVB) estimator $\hat{\mathbf{g}}_{\mathrm{MVB}}$ is the conditional expected value of the vector $\mathbf{g}$ on the condition that particular measurement data $\tilde{\mathbf{h}}$ were observed [6, 64, 79]. In reality Bayesian inverse techniques are realized by means of Monte Carlo simulations [6, 26, 27, 29, 64, 79, 80]. In this case, in order to obtain a posteriori probability distribution of vector $\mathbf{g}$ a statistical analysis is performed of a great number of outputs $\mathbf{h}$ of the mathematical model of indirect measurements computed for vectors $\mathbf{g}$ sampled from *a priori* probability distribution [6, 26, 27, 29, 64, 79, 80].

Apart from the presented numerical inverse procedures, also analytic methods of solving continuous inverse problem formulated in terms of Fredholm integral equation of the first kind are employed. Examples of such techniques are: method of moments [33] and integral transform method [34].

An useful technique for improving the numerical conditioning of the examined inverse problem is reduction of the complexity of the model by means of *e.g.* aggregation or selection of parameters [81].

In the case when sought-for quantities are known to satisfy certain constraints expressed by equations and inequalities one has to apply inverse methods based on constraint minimization techniques [82-85].

Global optimization techniques of are also employed for solving ill-posed and ill-conditioned inverse problems. Methods based on genetic algorithms are the most popular among these techniques [9-12].

Detailed presentation of particular numerical inverse techniques goes beyond the scope of the present review paper and can be the subject of another review paper.

7. Conclusion

The present review paper aimed at presenting the inverse problem in indirect measurement and – in particular – inverse problem formulated in terms of Fredholm integral equation of the first kind.

Many properties of real-world systems cannot be measured directly. Information on these properties can only be obtained as a result of indirect measurement which consists in direct measurement of other quantities followed by inference on sought-for quantities from directly measured quantities. The process of inference, called inverse problem in indirect measurement, is based on the mathematical model of the measurement system which is a mathematical relation between unknown quantities and measured quantities. In general, an inverse problem consists either in determining characteristics of a system under study, driven by controlled or known exciting signals, or in reconstructing exciting signals acting on a system whose internal characteristics are known. One can distinguish continuous and discrete inverse problems, depending on whether the measured and sought-for quantities are represented by functions or by vectors (tuples), respectively.

Very frequently nontrivial inverse problems in indirect measurements are ill-posed which means that they have no solution or the solution exists but is non-unique or unstable, *i.e.* the solution does not depend continuously on the measurement data. Due to measurement and round-off errors in measurement data many inverse problems do not have solutions in usual sense and hence generalized solutions, also known as quasisolutions, have to be considered which minimize the distance (metric) between the actual measurement data and the output of the mathematical model. In practice the instability of the solution (generalized solution) of an inverse problem manifests itself in high error amplification, referred to as ill-conditioning, which consists in that very small disturbances in the measurement data result in large disturbances in the result of inference. Consequently, the problem is quasi-underdetermined and its solution – quasi-nonunique. Ill-posedness and ill-conditioning result from the lack of information on sought-for quantities, carried by the measurement data. In other words, the condition of agreement of the sought-for solution with the measurement data is insufficient for determining the unique solution with satisfactory accuracy. Therefore, *a priori* knowledge about the space of admissible solutions has to be employed for solving such inverse problems.

The review paper was focused on a broad class of inverse problems formulated in terms of Fredholm integral equations of the first kind. Inverse problems of this type arise in many areas such as: computerized transmission tomography utilized in medicine, engineering and science, geophysical researches, spectrophotometric researches, particle sizing, digital signal processing and many other measurement, imaging and diagnostic techniques. In the paper ill-posedness and ill-conditioning of the original continuous version of these problems were analysed at the level of functional analysis. Smoothing properties of the Fredholm integral operator were indicated as a direct cause of ill-posedness and ill-conditioning. In the following parts of the review paper various techniques of discretization of Fredholm integral equation of the first kind were discussed. A concept of pseudosolution of discretized version of the problem and its ill-conditioning was explored based on singular value decomposition (SVD). Finally, selected groups of numerical methods of solving the discretized version of the inverse problem formulated in terms of Fredholm integral equation of the first kind were presented concisely.

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